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Engineering Notes

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Transmission Zeros in Structural Control with Collocated Multi-Input/ Multi-Output Pairs

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I. Introduction

IN ACTIVE vibration control of flexible structures with a single-input/single-output (SISO) collocated (dual) actuator/sensor pair, the transmission zeros exhibit the well-known property of *interlacing*, that is, the poles and zeros alternate along the imaginary axis (strictly if the structure is undamped, slightly in the left-half plane for a lightly damped structure). This property is the origin of the guaranteed stability of the so-called *low authority control (LAC)* strategies for active damping [1,2]. Besides, the transmission zeros coincide with the poles (natural frequencies) of a modified system which depends on the sensor configuration. For a displacement (velocity) sensor, the modified system is the constrained system where the degree of freedom (DOF) along which the control operates is blocked [3,4]. For a force sensor, the modified system is obtained by removing the contribution of the active member to the global stiffness matrix of the structure [2]. As the gain increases, the closed-loop poles start from the open-loop poles and those which remain at finite distance move on loops going asymptotically to the transmission zeros.

For displacement sensors, the extension of the SISO result to multi-input/multi-output (MIMO) collocated pairs has already been discussed in the literature [5] and, based on the fact that transmission zeros give identically zero output response, it has been *inferred* that, for MIMO systems, the transmission zeros are the eigenvalues of an associated constrained modes problem. To the authors' knowledge, however, no formal proof is available. This Note provides such a proof for an undamped structure; the two cases (force actuator, displacement sensor and displacement actuator, force sensor) are discussed separately.

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II. Force Actuator, Displacement Sensor

Consider an undamped MIMO system with n DOF equipped with m collocated actuator/sensor pairs:
structure:

$$M\ddot{x} + Kx = Bu \tag{1}$$

output:

$$y = B^T x \tag{2}$$

where M and K are the mass and stiffness matrices ($n \times n$). The $n \times m$ matrix B defines the topology of the actuator/sensor pairs and u and y are, respectively, the actuator input and sensor output vectors, both of size m. The same matrix B appears in both equations because of collocation.

A. Transmission Zeros

The transmission zeros of the system [6,7] are the values s_0 such that an input $u = u_0 e^{s_0 t}$ $(t \geq 0)$ applied from appropriate initial conditions x_0 produces a system response $x = x_0 e^{s_0 t}$ and a system output $y = 0$. From Eqs. (1) and (2),

$$\left(Ms_0^2 + K\right)x_0 = Bu_0 \tag{3}$$

$$B^T x_0 = 0 \tag{4}$$

or

$$\begin{pmatrix} Ms_0^2 + K & -B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = 0 \tag{5}$$

The values s_0 for which this system has a nontrivial solution are the transmission zeros of the system. Combining Eqs. (3) and (4), one gets

$$B^T \left(Ms_0^2 + K\right)^{-1} Bu_0 = 0 \tag{6}$$

s_0 is the eigenvalue and u_0 is the shape of the control input.

B. Asymptotic Values of the Closed-Loop Poles

Consider the output feedback

$$u = -gH(s)y \tag{7}$$

where $H(s)$ is a square matrix and g is a scalar parameter. The closed-loop eigenvalue problem is obtained by combining Eqs. (1), (2), and (7):

$$[Ms^2 + K + gBH(s)B^T]x = 0 \tag{8}$$

Theorem 8 of [8] extends the results for SISO systems to square (no. of inputs = no. of outputs) MIMO systems without feed-through: as $g \to \infty$, the finite eigenvalues of Eq. (8) coincide with the transmission zeros defined by Eq. (6), and this is for any form of $H(s)$ (not necessarily diagonal).

C. Alternate Proof Based on Woodbury Formula

In fact, for the particular case of collocated systems considered in this study, the identity between the asymptotic eigenvalues of Eq. (8) and those of Eq. (6) can be derived directly as follows: The Woodbury formula gives the inverse of a special block matrix. The case of interest here is

$$[A - VSV^T]^{-1} = A^{-1} - A^{-1}VTV^TA^{-1} \tag{9}$$

provided the matrix T is determined via

$$T^{-1} + S^{-1} = V^TA^{-1}V \tag{10}$$

The result stems from a report by Woodbury from 1950 and is given, in [9], p. 124 and [10], p. 51. It is simple to prove by direct multiplication. It generalizes the 1-dimensional Sherman–Morrison formula.

Applying this to Eq. (8) with $A = Ms^2 + K$ (system matrix), $V = B$ (topology of the control system) and $S(s) = gH(s)$ (control law), one finds that

$$[A + gBHB^T]^{-1} = A^{-1} - A^{-1}B[g^{-1}H^{-1} + B^TA^{-1}B]^{-1}B^TA^{-1} \tag{11}$$

This inversion will break down at the eigenvalues of s, leading to infinite values of the inverse. This will happen if the square bracket leads to infinite values, that is, for

$$\det[g^{-1}H^{-1} + B^TA^{-1}B] = 0 \tag{12}$$

At the limit for $g \to \infty$

$$\det[B^TA^{-1}B] = \det[B^T(Ms^2 + K)^{-1}B] = 0 \tag{13}$$

which is the characteristic equation of Eq. (6).

D. Evaluation of Transmission Zeros via Constrained System

Because the asymptotic solutions of the eigenvalue problem (8) do not depend on $H(s)$, they can be computed with $H(s) = I$. In this case, the transmission zeros are seen as the asymptotic solutions of

$$\lim_{g\to\infty}[Ms^2 + K + gBB^T]x = 0 \tag{14}$$

The matrix gBB^T is the contribution to the global stiffness matrix of a set of springs of stiffness g connected to all the DOF involved in the control. Asymptotically, when $g \to \infty$, the additional springs act as supports restraining the motion along the controlled DOF. Thus, the transmission zeros are the poles (natural frequencies) of the constrained system where the DOFs involved in the control are blocked. Because all the matrices involved in Eq. (14) are symmetrical and positive semidefinite, the transmission zeros are purely imaginary. Because blocking the controlled DOF reduces the total number of DOF by the number of control loops, the number of zeros is $2m$ less than the number of poles ($2n$).

III. Example 1

Consider the seven-story shear frame controlled in a decentralized manner with two independent and identical feedback loops as indicated in Fig. 1. Every actuator u_i applies a pair of forces equal and opposite between floor i and floor $i-1$, while the sensor $y_i = x_i - x_{i-1}$ measures the relative displacement between the same floors. The mass, stiffness, and B matrices are, respectively, $M = mI_7$,

$$K = k\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{15}$$

The natural frequency of mode l is given by (e.g., see [11])

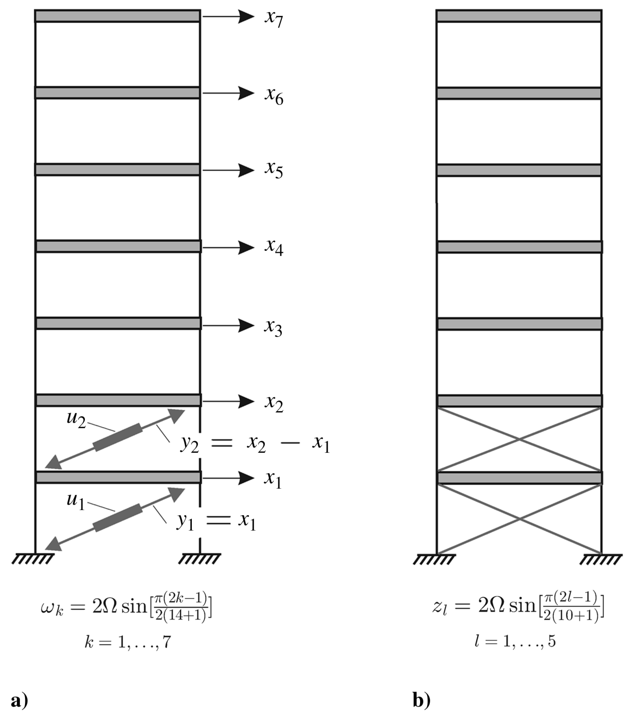


Fig. 1 a) Seven-story shear frame with two independent identical control loops (displacement sensor and force actuator pairs). b) Configuration corresponding to the transmission zeros $\pm z_k$.

$$\omega_l = 2\Omega \sin\left[\frac{\pi}{2}\frac{(2l-1)}{(2n+1)}\right], \qquad l = 1, \ldots, n \tag{16}$$

where $\Omega = \sqrt{k/m}$ and n is the number of stories. The feedback control law is

$$u = -gh(s)y \tag{17}$$

where g is the scalar gain and $h(s)$ is the scalar control law, common to all the loops.

According to the foregoing discussion, the transmission zeros are the natural frequencies of the system obtained by constraining (blocking) the first two floors. Equation (16) can therefore again be applied, after setting the number of stories to $n-2$. Figure 2 shows the root locus of a lead compensator

$$h(s) = \frac{1 + Ts}{1 + \alpha Ts} \qquad (\alpha > 1) \tag{18}$$

while Fig. 3 shows the root locus for a positive position feedback as in [12]

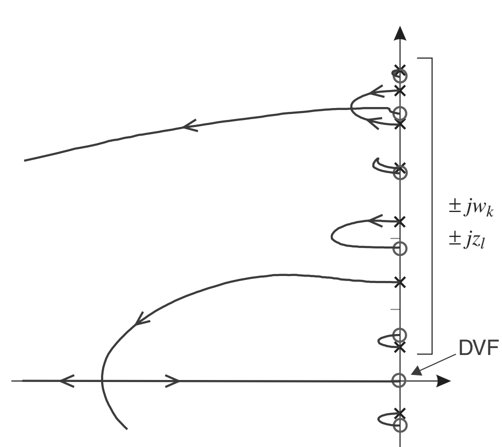


Fig. 2 Root-locus plot for the DVF.

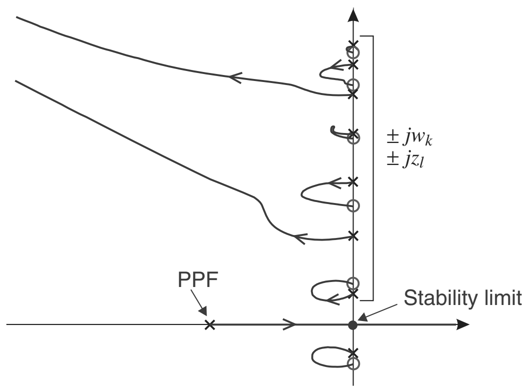


Fig. 3 Root-locus plot for the positive position feedback (PPF) (19).

$$h(s) = \frac{-1}{1 + \tau s} \tag{19}$$

In generating Fig. 2, the zero of Eq. (18) was chosen very close to the origin and the pole very far on the negative real axis, so that the lead compensator behaves as a direct velocity feedback (DVF). In Fig. 3, τ is taken such that $\tau\omega_1 = 0.5$. (Note that, in both root-locus plots, the scale has been magnified on the real axis, for clarity purpose, so that the plots do not reflect the exact shape of the various loops.) The two root-locus plots share the same asymptotic values for $g = 0$ (open-loop poles) and for $g \to \infty$ (transmission zeros); for $0 < g < \infty$, the plots depend on the control law.

Unlike in the SISO case, it is not possible to draw the entire root-locus plot from the knowledge of the open-loop poles and zeros alone. The root locus for MIMO systems does not follow the same rules as for SISO systems, and it is not possible to know a priori the connectivity between the various poles and zeros. The asymptotic properties of the closed-loop poles of a MIMO, linear quadratic regulator (LQR) controller have been analyzed in [13]. The present discussion has been restricted to undamped structures; the effect of damping on transmission zeros has been analyzed in [14].

IV. Displacement Actuator, Force Sensor

Another frequent configuration met in active vibration control involves collocated displacement actuator and force sensor pairs [2]. Such a system can be represented by

structure:

$$M\ddot{x} + Kx = BK_a u \tag{20}$$

output:

$$y = K_a(B^T x - u) \tag{21}$$

In this equation, K is the global stiffness matrix, including all the active members, B defines the topology of the control system, K_a is the diagonal stiffness matrix of the active members, and u is the vector of unconstrained expansion of the actuators. These equations apply to a wide variety of actuators such as piezoelectric, magnetostrictive, thermal, shape memory alloy (SMA), ball screw, ... In the output equation, $B^T x$ is the total elongation of the active members and $(B^T x - u)$ is the elastic extension; Eq. (21) simply states that the output force of every sensor is the product of its stiffness and the elastic extension. As compared to the previous case of the force actuator and displacement sensor, the situation is slightly different because of the feedthrough component in Eq. (21) which makes theorem 8 of [8] not directly applicable.

A. Transmission Zeros

Proceeding as before, the transmission zeros of the system are the values s_0 such that an input $u = u_0 e^{s_0 t}$ $(t \geq 0)$ applied from appropriate initial conditions x_0 produces a system response $x = x_0 e^{s_0 t}$ and a system output $y = 0$. From Eqs. (20) and (21),

$$\left(Ms_0^2 + K\right)x_0 = BK_a u_0 \tag{22}$$

$$B^T x_0 - u_0 = 0 \tag{23}$$

or

$$\begin{pmatrix} Ms_0^2 + K & -BK_a \\ B^T & -I \end{pmatrix}\begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = 0 \tag{24}$$

Upon eliminating u_0, one gets

$$\left(Ms_0^2 + K - BK_a B^T\right)x_0 = 0 \tag{25}$$

The condition for a nontrivial solution is

$$\det\left(Ms_0^2 + K - BK_a B^T\right) = 0 \tag{26}$$

In this equation, $BK_a B^T$ is the contribution of the active members to the global stiffness matrix, so that Eq. (26) is the characteristic equation of the system after removing the contribution of the active members to the global stiffness matrix. Thus, the transmission zeros are the poles (natural frequencies) of the system where the contribution of the active members to the stiffness matrix has been removed. Because removing the active member does not change the number of DOF, the number of zeros is equal to the number of poles.

B. Asymptotic Values of the Closed-Loop Poles

Now, let us consider the output feedback

$$u = gH(s)y \tag{27}$$

where $H(s)$ is a square matrix and g is a scalar gain (note the positive feedback in this case). Combining Eqs. (27) and (21),

$$u = (I + gHK_a)^{-1} gHK_a B^T x \tag{28}$$

and substituting in Eq. (20), one gets

$$[Ms^2 + K - BK_a(I + gHK_a)^{-1} gHK_a B^T]x = 0 \tag{29}$$

This is the closed-loop eigenvalue problem. The asymptotic values are, respectively, for $g = 0$, the open-loop poles (natural frequencies of the system including the active members) and, for $g \to \infty$, $(I + gHK_a) \sim gHK_a$,

$$(Ms^2 + K - BK_a B^T)x = 0 \tag{30}$$

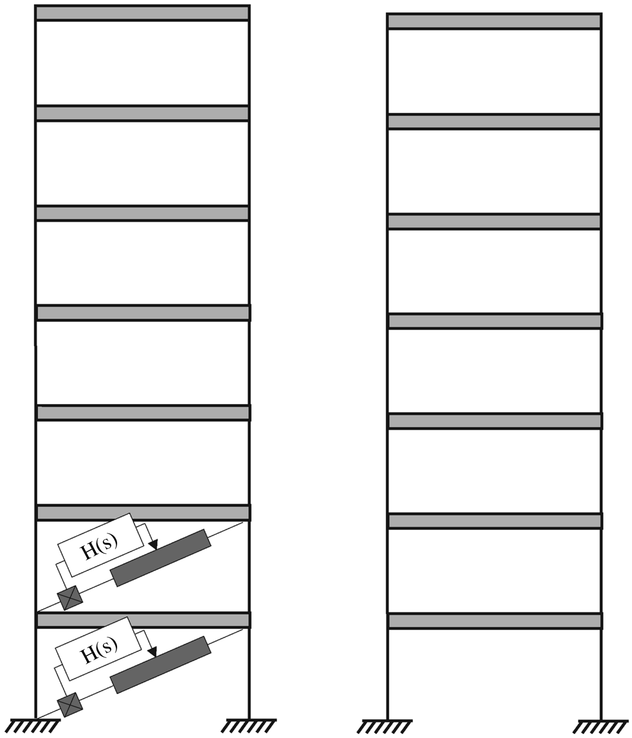
the characteristic equation of which is Eq. (26). Thus, asymptotically, as $g \to \infty$, the finite eigenvalues coincide with the transmission zeros. This result generalizes earlier ones obtained in the particular case of identical integral force feedback (IFF) loops [15]. An experimental verification with a truss controlled with three decentralized active tendons can be found in [2,16].

V. Example 2

Consider again the seven-story shear frame of the previous example (Fig. 4), with active members of stiffness k_a acting in the first two floors ($k_a/k = 5$ has been used in the numerical example). Figure 5 shows the root-locus plot for the IFF controller

$$h(s) = g/s \tag{31}$$

It is worth noting that 1) the zeros of this example are identical to the poles of the previous example; 2) the poles and zeros alternate on the imaginary axis in this case, but this is not a general rule in the MIMO case (e.g., see Fig. 14.14 of [2]), and 3) the root locus of Fig. 5 is different from that corresponding to a SISO system with the same pole-zero pattern (as in the previous example).



$$z_k = 2\Omega \sin[\frac{\pi(2k-1)}{2(14+1)}]$$

$$k = 1, \ldots, 7$$

a) b)

Fig. 4 a) Seven-story shear frame with two independent identical control loops (force sensor and displacement actuator pairs). b) Configuration corresponding to the transmission zeros $\pm z_k$.

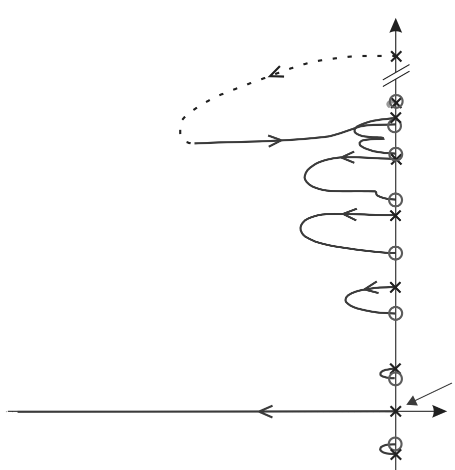


Fig. 5 Root-locus plot for the IFF.

VI. Conclusions

This Note investigated the transmission zeros of an undamped structure with a set of collocated (dual) actuator/sensor pairs. The transmission zeros are purely imaginary and coincide with the poles (natural frequencies) of a modified structure, the topology of which depends on the nature of the actuator/sensor pairs. In the case of displacement (velocity) sensors, it is obtained by blocking the DOF along which the displacement sensors operate. In the case of force sensors, the contribution of the active members (where force sensors are placed) to the global stiffness matrix is canceled. Note, however, that the interlacing property of the SISO case does not hold any longer. As the gain increases $0 \leq g < \infty$ the closed-loop poles start from the open-loop poles and those which remain at finite distance move asymptotically to the transmission zeros; the precise trajectory depends on the control law but the asymptotic values do not. However the full root locus cannot be plotted from the open-loop poles and transmission zeros alone as in the SISO case. The discussion has been illustrated with a seven-story shear frame example controlled in a decentralized manner with various control strategies.

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